ON THE MOTION OF A RIGID BODY CONTAINING CAVITIES FILLED WITH A VISCOUS FLUID

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Considered is the problem of the motion of a rigid body containing a cylindrical (plane motion) or a spherical cavity filled with a viscous fluid. In the case of small oscillations of the body with the cylindrical cavity, the solution is obtained by the operational method. For the general case, a method is proposed which is applicable for a very viscous fluid when the integro-differential equations of motion are reduced to the ordinary equations with a small parameter at the higher derivative. The results are applicable in the theory of rotation of a body about a fixed point.

1. The investigation of particular cases of motion of bodies containing cavities with viscous fluids is presented in [1 and 2] and elsewhere, often as an example of operational methods. In [3] the oscillations of a body were considered under the condition that the Reynolds number is large, and the investigation of fluid motion utilized an approximation of the boundary layer. In the following is considered the problem of body motion containing a viscous fluid with such cavities for which the solution of a certain nonstationary hydrodynamical problem is known.

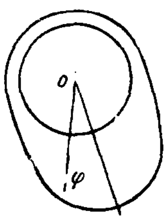

Fig. 1

First is considered the simplest case of the plane problem. Let there be given a rigid body containing a cavity in the form of a circular cylinder the axis of which coincides with the rotation axis of the body (Fig.1). The case when the rotation axis does not coincide with the central axis does not essentially complicate the problem. Let it be required to solve the problem of small oscillations for such a body in the presence of a restoring moment proportional to the deviation. Considered will be only the case of zero initial velocity of the fluid. In the general case of an arbitrary initial distribution of velocities, it is necessary only to find the superposition of the following solution and the solution of the problem of body motion in the absence of external forces and an arbitrary initial distribution of fluid velocities in view of the linearity of the considered problem [2].

For the following, it is necessary to know the solution of a problem of viscous fluid motion in a cylinder when the cylinder, initially at rest, is instantly brought to a constant angular velocity ω . This solution will be obtained in accordance with [2]. The fluid particle trajectories may be considered circular. It then follows that $v_r = 0$ and $\partial v_\varphi / \partial \varphi = 0$, where v_r and v_φ are the fluid particle velocity components along the radius and the perpendicular to the radius, φ is the central angle. The Navier-Stokes equation is then of the form

$$\frac{\partial v_\varphi}{\partial t_1} = \nu \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2} \right)$$

The solution is sought for the initial and boundary conditions

$$v_\varphi = 0, \quad \text{for } t = 0, \qquad v_\varphi = \omega a \quad \text{for } r = a$$

Application of the Laplace transform method leads to the following expression for the fluid velocity

$$v_\varphi (r, t_1) = \omega a \left[\frac{r}{a} + 2 \sum_{k=1}^{\infty} \exp \left(- \nu \frac{\lambda_k^2}{a^2} t_1 \right) \frac{J_1(ra^{-1}\lambda_k)}{\lambda_k J_1' (\lambda_k)} \right]$$

Here ν is the kinematical coefficient of viscosity, a is the cylinder radius, λ_k are the roots of the Bessel function of the first kind.

Utilizing the expression for the velocity of deformation in polar coordinates, it can be established that the force of viscosity for circular motion of the fluid is given by Formula

$$\tau = \mu \left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} \right)$$

where μ is the dynamical coefficient of viscosity.

The expression for the moment of fluid forces acting on the cylinder walls is obtained from the last two formulas in the following form

$$L_1 = L\omega = 4\pi\mu a^2 \omega \sum_{k=1}^{\infty} \exp \left(- \nu \frac{\lambda_k^2}{a^2} t_1 \right) \tag{1.1}$$

Let J be the body moment of inertia, $M\varphi$ the moment of the rotating force; then the equation of motion for the body is

$$J \frac{d^2\varphi}{dt_1^2} + M\varphi = N$$

Here N is the moment due to the fluid acting on the body. The expression for it is easily found by means of the Duhamel Integral if the solution is known for the corresponding hydrodynamical problem (1.1) and the angular velocity $\omega(t_1)$ is given. Thus,

$$N = - \int_0^{t_1} \omega' (\tau) L (a, t_1 - \tau) \, d\tau$$

In the present case, the value of the angular velocity $\omega (t_1) = d\varphi / dt_1$ is unknown, and therefore, the equation of motion is integro-differential. Utilizing the non-dimensional variables

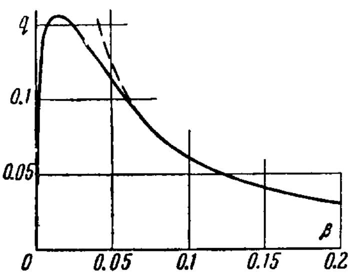

Fig. 2

$$t = t_1 \left(\frac{M}{J} \right)^{1/2}, \qquad \beta = \frac{\nu}{a^2} \left(\frac{J}{M} \right)^{1/2}, \qquad \gamma = \frac{\mu a^2}{\sqrt{MJ}}$$

the equation of motion is transformed into the following form

$$\frac{d^2\varphi}{dt^2} + \varphi = -4\pi\gamma \int_0^t \frac{d^2\varphi}{dt^2} \sum_{k=1}^{\alpha} \exp \left[- \beta\lambda_k^2 (t - \tau) \right] d\tau \tag{1.2}$$

This linear equation will be solved by the methods of operational calculus. Let Φ denote the Laplace transform

$$\Phi(p) = \int_0^\infty \varphi(t)\, e^{-pt}\, dt = [\varphi(t)]^*$$

Applying the Laplace transform to both sides of (1.2) and by changing the order of integration analogous to [2], the formula

$$[\varphi''(t)]^* + \Phi = -4\pi\gamma\, [\varphi''(t)]^* \sum_{k=1}^{\infty} (p + \beta\lambda_k^2)^{-1}$$

is obtained.

Hence, utilizing the known equality $[\varphi'(t)]^* = p\Phi(p) - \varphi(0)$ and the initial conditions $\theta(0) = \varphi_0 = \text{const}$ and $\varphi'(0) = 0$, we obtain in accordance with the inverse transformation

$$\varphi(t) = \frac{\varphi_0}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{pt} \left(1 + 8\sigma \sum_{k=1}^{\infty} \frac{1}{p\beta^{-1} + \lambda_k^2}\right) \left[1 + p^2 \left(1 + 8\sigma \sum_{k=1}^{\infty} \frac{1}{p\beta^{-1} + \lambda_k^2}\right)\right]^{-1} dp$$

$$\left(\sigma = \frac{\pi\gamma}{2\beta} = \frac{\pi\rho a^4}{2J} = \frac{J_1}{J}\right) \qquad (1.3)$$

Here δ is a small positive quantity. This integral contains the characteristic nondimensional quantities β and σ, where ρ is the density of the fluid. The first one characterizes the relationship of the viscous forces to the external forces. The second characterizes the relationship between the moments of inertia of the hardened fluid J_1 and the body J. In order to determine the motion, it is necessary to compute the integral (1.3) which can be done for each specific pair of values of β and σ. It can be easily seen that the subintegral expression has a denumerable set of poles on the negative real axis and two poles at the complex conjugate points to the left of the imaginary axis.

In the important practical case when the mass of the fluid is a smaller part of the mass of the whole body, the absolute quantity of the real part of the complex conjugate roots is significantly smaller than the modulus of the smallest root of the denominator on the real axis. This means that the corresponding motion has the slowest decay and, therefore, its analysis is most interesting in studying the oscillations of a body containing a liquid.

For $\sigma = 1.6$ and for the various values of β the real parts of the complex roots $q = -\mathrm{Re}\, p_k$ were evaluated as well as the denominator of Equation (1.3) which is equal to the logarithmic decrement of the oscillation decay. The real and imaginary parts were separately set equal to zero in the denominator of (1.3). The results of the computations are shown in Fig.2 (solid curve). From this plot it may be generally concluded that the decay is maximum for a certain finite value of β and that it is zero for $\beta = 0$ when the fluid is ideal and does not participate in the motion of the body, as well as for $\beta = \infty$ when the fluid rotates along with the cavity walls as a rigid body.

2. In the following will be considered the limiting case of a fluid filled body when β is large, i.e. when the fluid rotates almost as a rigid body. Application of double integration by parts in the right-hand side of (1.2) lead to Equation

$$\varphi'' + \varphi = -4\pi\gamma \left(\varphi'' \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} - \varphi''' \frac{1}{\beta^2} \sum_{k=1}^{\infty} \frac{1}{\lambda_k^4} + \right.$$

$$\left. + \frac{1}{\beta^3} \int_0^t \varphi^{\mathrm{IV}} \sum_{k=1}^{\infty} \frac{1}{\lambda_k^6} \exp\left[-\beta\lambda_k^2 (t-\tau)\right] d\tau \right) \qquad (2.1)$$

The solution of Equation (2.1) will be sought in the class of functions bounded along with the first four derivatives. Apparently, the last term in (2.1) is $O(\beta^{-3})$ and therefore, it may be neglected when compared with the foregoing term for sufficiently large β. The validity of this assumption can be demonstrated in the derived, shortened equation (from (2.1)) will have solutions with bounded four derivatives.

The summations contained in (2.1) are known in the theory of Bessel functions. After the replacements of the summations by their numerical values and dropping of the integral term, the following equation is obtained:

$$-\varepsilon_1\varphi''' + (1+\sigma)\varphi'' + \varphi = 0 \qquad (\varepsilon_1 = {}^1/_{12}\sigma\beta)$$

Here ε_1 is a small quantity. Let $t' = t/\sigma$ be a new independent variable and $\varepsilon = \varepsilon_1/(1+\sigma)^{3/2}$.

The last equation becomes

$$-\varepsilon\varphi''' + \varphi'' + \varphi = 0 \tag{2.2}$$

The general solution of Equation (2.2) is of the form

$$\varphi = A\exp(-{}^1/_2\varepsilon t')\sin(t'+B) + C\exp(t'/\varepsilon) \tag{2.3}$$

According to the condition for boundedness of the solution at infinity the constant C must be equal to zero. Thus, for large β the body performs decaying oscillations with the logarithmic decrement of decay equal to $q = \sigma/24\beta\,(1+\sigma)^{5/2}$.

Fig. 2 indicates the dependence of q upon β (dotted curve) for $\sigma = 1.6$, i.e. for the same σ as in the preceding section.

Formula (2.3) (for $C = 0$) can also be obtained directly from (1.3) by computing the real and imaginary parts of the complex conjugate roots (the remaining roots tend to $-\infty$ for $\beta \to \infty$). Indeed, the expansion of the summations contained in (1.3) in powers of $p/\beta\lambda_k^2$ according to the Newton's binomial theorem leads to the following two-term representation

$$\sum_{k=1}^{\infty}\left(\frac{p}{\beta}+\lambda_k^2\right)^{-1} = \sum_{k=1}^{\infty}\frac{1}{\lambda_k^2} - \frac{p}{\beta}\sum_{k=1}^{\infty}\frac{1}{\lambda_k^4} + \ldots = \frac{1}{8} - \frac{p}{\beta}\frac{1}{96} + \ldots$$

The equation for determination of the roots of the denominator in the first approximation is of the form

$$1 + p^2\left(1+\sigma-\frac{\sigma}{12}\frac{p}{\beta}\right) = 0$$

By means of sequential computation of the zeroth and the first approximations the above derived formula for $q = -\mathrm{Re}\,p$ is easily obtained from this equation. Formula (2.3) is likewise obtained after computation of the residues in the roots of the subintegral expressions in (1.3). The last method of solution is, unfortunately, not applicable in nonlinear problems.

3. In the following will be considered the motion of a body having a spherical cavity filled with a viscous fluid the center of which coincides with the center of inertia of the body and with the point of support. It will be shown that this problem can also be solved analogously for a cavity of any arbitrary form if only the hydrodynamical problem of fluid motion is known when the body begins rotating with a constant angular velocity, the solution being linearly dependent upon it. For the case of the spherical cavity such a solution of the hydrodynamical problem is known for small Reynolds number [2].. In this case, the fluid particles move along parallel circles. This case will be considered in the sequel. The formula for the moment of forces applied by the fluid to the body when it begins rotating with constant angular velocity ω is [2]

$$\omega L = \frac{16}{3}\pi\mu\omega a^3\sum_{k=1}^{\infty}\exp\left(-\nu\frac{\delta_k^2}{a^2}t_1\right)$$

where a is the radius of the sphere, δ_k are the roots of Equation $\delta = \tan\delta$.

In the following will be considered only the case with zero external forces (Euler case), the equation of motion for which is in vector form

$$\frac{d\mathbf{K}_1}{dt_1} = -\int_0^{t_1} \frac{d\mathbf{\Omega}_1}{d\tau} L(t_1 - \tau)\, d\tau$$

Here d/dt_1 is the absolute time derivate, $\mathbf{K}_1$ and $\mathbf{\Omega}_1$ are the momentum and the instantaneous angular velocity vectors. If both parts of this equation are divided by $\omega_0{}^2 J$ where ω_0 is the initial angular velocity of the body, J is the moment of inertia about some axis, and if the following nondimensional quantities are introduced

$$t = t_1\omega_0, \qquad \beta = \nu / a^2\omega_0, \qquad \gamma = \mu a^3 / J\omega_0$$

$$A_0 = A_1 / J, \quad B_0 = B_1 / J, \quad C_0 = C_1 / J, \quad \mathbf{K} = \mathbf{K}_1 / J\omega_0, \quad \mathbf{\Omega} = \mathbf{\Omega}_1 / \omega_0$$

then the equation of motion in nondimensional variables is of the form

$$\frac{d\mathbf{K}}{dt} = -\frac{16}{3}\pi\gamma \int_0^t \frac{d\mathbf{\Omega}}{d\tau} \sum_{k=1}^{\infty} \exp\left(-\beta\delta_k{}^2 (t - \tau)\right) d\tau \tag{3.1}$$

Let as in the previous case β denote a large parameter and let it be required to find bounded solutions of (3.1) satisfying the given initial conditions for $\mathbf{\Omega}$. Double integration by parts, similar to the preceding integration, and the elimination of the higher order integral term leads to Equation

$$\frac{d\mathbf{K}}{dt} = -\frac{16}{3}\pi\gamma \left(\frac{d\mathbf{\Omega}}{dt}\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{\delta_k{}^2} - \frac{d^2\mathbf{\Omega}}{dt^2}\frac{1}{\beta^2}\sum_{k=1}^{\infty}\frac{1}{\delta_k{}^4}\right) \tag{3.2}$$

For the solution of the last equation, it is convenient to introduce a body-fixed moving coordinate system with axes along the principal axes of inertia. The transformation from the absolute derivatives to the derivatives in the moving coordinate system (which will be denoted by d'/dt or a prime) is given by Formula [4]

$$\frac{d\mathbf{h}}{dt} = \frac{d'\mathbf{h}}{dt} + \mathbf{\Omega} \times \mathbf{h}$$

where $\mathbf{h}$ is an arbitrary vector.

Equation (3.2) then becomes

$$\frac{d'\mathbf{K}}{dt} + \mathbf{\Omega} \times \mathbf{K} = -D\,\frac{d'\mathbf{\Omega}}{dt} + \varkappa\left(\frac{d'^2\mathbf{\Omega}}{dt^2} + \mathbf{\Omega} \times \frac{d'\mathbf{\Omega}}{dt}\right)$$

$$\left(DJ = \frac{8\pi\rho a^5}{15},\ \varkappa \approx \frac{0.028D}{\beta}\right)$$

Here DJ is the moment of inertia of the hardened liquid, $\varkappa$ is a constant small parameter. In scalar form we have

$$\begin{aligned} Ap' + (C - B)\,qr &= \varkappa\,(p'' + qr' - rq') \\ Bq' + (A - C)\,rp &= \varkappa\,(q'' + rp' - pr') \\ Cr' + (B - A)\,pq &= \varkappa\,(r'' + pq' - qp') \end{aligned} \tag{3.3}$$

Here $A = A_0 + D$, $B = B_0 + D$, $C = C_0 + D$ and p, q, r are the projections of the angular velocity on the moving axes.

In the following will be considered only the case of the axisymmetrical body when $A = B$. Also if $C > A$ and $\varkappa = 0$, i.e. for a hardened liquid, the solution of Equation (3.3) is

$$r = r_0, \qquad p = P_0\cos(\omega t + \alpha), \qquad q = P_0\sin(\omega t + \alpha), \qquad \omega = (C - A) / Ar_0 \tag{3.4}$$

Here P_0, r_0 and α are certain constants. The motion described by Formulas (3.4) is the regular precession. In the following for $\varkappa$ small but finite, the solution will be sought in the form of (3.4) where r_0, P_0 and ω are replaced by r, P and ω which will be regarded as the slowly varying functions of time. Indeed, for sufficiently small $\varkappa$ during a short segment of time (compared to $2\pi/\omega$) the motion of the body must be close to regular precession, i.e. r and p must change a small amount. From Equations (3.3) in the zero approximation, it follows that

$$P^2 = (l^2 - C^2r^2) / A^2$$

where l^2 is a constant of moments. But since l^2 is independent of time in view of the conservation of momentum law, this equation is valid for all instants of time. It yields the relationship between P and r. The equation for r, however, is obtained by substitution of (3.4) into the last equation in (3.3), and by dropping of terms of order higher than first.

$$Cr' - \varkappa (C - A) P^2 r / A = 0$$

Note that in the derivation of the last relationship the term $\varkappa r''$ was neglected which is of second order since the desired solution is slowly varying with time. Evaluation of the quadrature leads to the Formula

$$(l^2 - C^2r^2)^{1/2} r^{-1} = E \exp [- \varkappa (C - A) l^2 t / A^3 C] \qquad (3.5)$$

where E is a constant of integration. Formula (3.5) was derived under the assumption that $C > A$, i.e. that the body is flattened. For $C < A$, i.e. for an elongated body, the solution cannot be sought in the form (3.4) since then $\omega < 0$ which has no physical meaning. Instead, the following solution of Equations (3.3) should be utilized:

$$r = r_0, \qquad p = P_0 \sin (\omega t + \alpha), \qquad q = P_0 \cos (\omega t + \alpha), \qquad \omega = (A - C) r_0 / A$$

Analogous considerations lead again to Formula (3.5). Thus, this formula gives the asymptotic solution of the formulated problem in all cases. It shows that for $C = A$, the rotation does not change with time (dynamic symmetry), and for $C > A$ the body eventually rotates about the z-axis ($r \to l/C$). For $C > A$ Equation (3.5) shows that $r \to 0$ and $P \to l/A$ and the angular velocity of precession tends to zero, i.e. there exists a limiting position of the axis of revolution which lies in the xy plane.

Also, it is worth noting, that (as can be seen from (3.5)) for sufficiently small $\varkappa$ the derived equation varies slowly with time which proves the validity of the assumptions made.

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